Technical note on Friedgut's sharp threshold theorems for non-uniform hypergraphs

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1 Introduction

The purpose of this technical note is to generalize Friedgut's [1], [2] results on sharp threshold to cover non-uniform random hypergraphs. The motivation comes from joint work with Jakub Kozik on 2-colorability of random non-uniform hypergraphs.

For brevity, we will refer to (possibly non-uniform) hypergraphs as graphs.

1.1 Random (hyper)graphs

For a function $\mathcal{M} : \mathbb{N} \to [0,1]$, let $\mathcal{H} = H(n;\mathcal{M})$ be a random graph with a set of vertices $V = \{1, 2, \ldots, n\}$ and a (random) set of edges, where every possible subset $S \subseteq V$ is included independently with probability $\mathcal{M}(|S|)$. For convenience, let $p_k = \mathcal{M}(k)$ and $q_k = 1 - p_k$. Let $\mu = \mu_{\mathcal{H}}$ be the measure corresponding to our model.

Given graph H, let $H^{(k)}$ be its subgraph induced on the edges of size k. Let

$$\mathcal{P}_H = \prod_k p_k^{|H^{(k)}|}$$

i.e. $\mathcal{P}_H$ corresponds to the probability of appearance of a specific copy of H. Additionally let's define

$$\mathcal{Q}_H = \prod_k q_k^{|H^{(k)}|}$$

Let $E(H)$ denote the expected number of copies of H in a random graph, then

$$E(H) = \binom{n}{v(H)} \mathcal{P}_H \frac{v(H)!}{|Aut(H)|}$$

where $Aut(H)$ is the group of the autmorphisms of H. When $|H|$ is bounded by a constant (which is usually the case in the following sections), then the following, asymptotically equivalent function is more convenient to work with

$$D(H) = n^{v(H)} \mathcal{P}_H$$

1.2 Influence

Let f be a characteristic function for a symmetric, monotonne family of graphs. For an edge e, let $I_e(f)$ be the probability that f changes value, when edge e is 'flipped' in a random graph. Let $I^{(k)}$ denote the sum of influences of edges of arity k and $I = \sum_k I^{(k)}$.

1.3 Main result

We were able to extend the main result from [1] to the non-uniform setting, under minor additional assumption.

Theorem 1.1 (Main theorem)**.**
There exists a function $k(\varepsilon, c)$, s.t. for all $c > 0$, any n and any monotonne symmetric family of graphs $\mathcal{A}$ on n vertices the following holds: If there exist a sequence (p_k) where each p_k is proportional to a rational power of n and $\sum_k p_k \cdot I^{(k)} \leq c$, then for every $\varepsilon > 0$ there exists a monotone symmetric family $\mathcal{B}$ s.t. $||\mathcal{B}|| \leq k(\varepsilon, c)$ and $\mu(\mathcal{A} \triangle \mathcal{B}) \leq \varepsilon$. Furthermore, the minimal graphs in $\mathcal{B}$ are balanced.

Assumption about p_k being a rational power of n is unfortunately necessary for the proof in the non-uniform setting. In the original paper by Friedgut [1] this is a corollary of the existence of small subgraph H with $E(H) = \Theta(1)$. In our setting, this assumption is needed for the following: Let f be a characteristic function of a graph property and $g(G) = f(G \cup H)$, where H is small, strictly balanced graph with $E(H) = \Theta(1)$. Then $\sum_k p_k \cdot I^{(k)}(f) = \mathcal{O}(1) \implies \sum_k p_k \cdot I^{(k)}(g) = \mathcal{O}(1)$.

We also successfully extended the theorem which guarantees the existence of a *special graph* whose appearance indicates the property almost surely.

Theorem 1.2 (Special graph)**.**
Let $0 < \alpha < 1$. There exist functions $B(\varepsilon, c), b_1(\varepsilon, c), b_2(\varepsilon, c)$ s.t. for all $c > 0$, any n and any monotone symmetric family of graphs $\mathcal{A}$ on n vertices the following holds: If there exists a sequence (p_k) where each p_k is proportional to a rational power of n, $\sum_k p_k \cdot I^{(k)} \leq c$ and $\alpha < \mu(\mathcal{A}) < 1 - \alpha$, then for every $\varepsilon > 0$ there exists a graph M with the following properties:

- *M is balanced*
- *$b_1 < E(M) < b_2$*
- *$|M| \leq B$*
- *Let $\mathcal{P}[\mathcal{A}|M]$ denote the probability that a random graph belongs to $\mathcal{A}$ conditioned on the appearance of a* specific *copy of M. Then*

$$\mathcal{P}[\mathcal{A}|M] \geq 1 - \varepsilon$$

Due to the symmetry of the space of random graphs, instead of conditioning on a specific copy of M, we can condition on the appearance of any copy of M.

2 Preliminaries

2.1 Thresholds

Let $\mathcal{A}$ be a family of graphs and $\mathcal{A}_n$ be its restriction to graphs on n vertices. The property is monotone if $H \in \mathcal{A} \implies G \in \mathcal{A}$ for any $H \subseteq G$. The property is symmetric if it is invariant under automorphisms.

The property has *sharp threshold* if there exists a series of random model parametrizations $\mathcal{M}_n$ s.t. for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mu_{\mathcal{M}}(\mathcal{A}_n) = \begin{cases} 1, \text{ if } \mathcal{M}(k) > \mathcal{M}_n(k) \cdot (1+\varepsilon) \ \forall_k \\ 0, \text{ if } \mathcal{M}(k) < \mathcal{M}_n \cdot (1-\varepsilon) \ \forall_k \end{cases}$$

where $\mathcal{M} \cdot (1 \pm \varepsilon)$ denotes the function $\mathcal{M}$ uniformly increased (decreased) multiplicatively by ε.

If a property does not have a sharp threshold, then it has a *coarse threshold*.

2.2 Balanced (hyper)graphs

A k-uniform hypergraph $\mathcal{H}$ is *balanced* if its average degree is not smaller than the average degree of any of its subgraphs. It is *strictly balanced* if the average degree of its every proper subgraph is strictly smaller.

In the context of random k-uniform graphs, these conditions can be interpreted as follows. For a balanced k-graph, the expected number of its copies in a random hypergraph is asymptotically not larger than the expectation of any of its subgraphs. Moreover, for strictly balanced k-graphs, the expected number of its subgraphs is strictly larger.

To avoid unnecessary complications, to extend these definitions to the non-uniform setting, we will use the second approach. Note, that under such definition, the notion of (strict) balancedness depends on a particular distribution $\widehat{H}(n; \mathcal{M})$.

2.3 Fourier base

We'll express our function f in an orthonormal basis with respect to μ. Let $U_\emptyset = 1$. For any edge e of arity k, let:

$$U_e(H) = \begin{cases} -\sqrt{q_k/p_k} & \text{if } e \in H, \\ \sqrt{p_k/q_k} & \text{if } e \notin H. \end{cases}$$

For any graph R:

$$U_R(H) = \prod_{e \in R} U_e(H)$$

The second condition - base functions for non-trivial graphs being products of functions for their components, the choice of these functions is (up to the sign) forced.

2.4 Fourier expansion and coefficients

The Fourier expansion in such basis is given by

$$f = \sum_{H} \hat{f}(H) U_H$$

where $\hat{f}(H)$ is a Fourier coefficient of a subgraph H, given by

$$\hat{f}(H) = \langle f, U_H \rangle = \int f(R) \cdot U_H(R)\, dR = \sum_{R} f(R) \cdot U_H(R) \mu(R)$$

2.5 Russo, Margulis

Lemma 2.1 (Russo, Margulis)**.**

$$\sum_{k} \frac{\partial \mathbb{P}(\mathcal{A})}{\partial p_k} = I$$

Proof. Let N be the number of all possible edges and $\rho_1, \rho_2, \ldots, \rho_N$ be the probabilities of appearance of corresponding edges. We can express both $\mathbb{P}(\mathcal{A})$ and I more explicitly as $\mathbb{P}_{\rho_1,\rho_2,\ldots,\rho_N}(\mathcal{A})$ and $I_{\rho_1,\rho_2,\ldots,\rho_N}$. Let e_j be an edge of arity k. Let

$$X := \{H : f(H) = 1 \text{ and } f(H+e) = 1\} \text{ and } Y := \{H : f(H) = 0 \text{ and } f(H+e) = 1\}$$

Then, $\mathbb{P}(\mathcal{A})$ can be expressed as

$$\mathbb{P}_{\rho_1,\rho_2,\ldots,\rho_N}(\mathcal{A}) = \mathbb{P}_{\rho_1,\ldots,\rho_{j-1},\rho_{j+1},\ldots,\rho_N}(X) + \mathbb{P}_{\rho_1,\ldots,\rho_{j-1},\rho_{j+1},\ldots,\rho_N}(Y) \cdot \rho_j$$

$$= \mathbb{P}_{\rho_1,\ldots,\rho_{j-1},\rho_{j+1},\ldots,\rho_N}(X) + I_{e_j} \cdot \rho_j \Rightarrow \frac{\partial \mathbb{P}(\mathcal{A})}{\partial \rho_j} = I_{e_j}$$

Identifying probabilities corresponding to edges of common arity yields desired result. $\square$

2.6 e-dependent functions

Let e be an edge of arity k, then

$$f_e(H) = \begin{cases} q_k\left(f(H) - f(H \oplus e)\right), \text{ if } f(H) = 1 \\ p_k\left(f(H) - f(H \oplus e)\right), \text{ if } f(H) = 0 \end{cases}$$

Such function is highly related with Fourier coefficients of subgraphs containing e and its influence i.e. I_e.

Lemma 2.2 (e-dependent properties)**.**

$$\hat{f}_e(H) = \begin{cases} \hat{f}(H), \ \textit{if } e \in H \\ 0, \ \textit{if } e \notin H \end{cases}$$

Proof. Note that $f_e(H)$ is non-zero iff $f(H) \neq f(H \oplus e)$, i.e. when H is *flipping*. Moreover, when H is flipping, then

$$f_e(H) = \begin{cases} q_k, & \text{if } e \in H \\ -p_k, & \text{if } e \notin H \end{cases}$$

Let's first examine the case where $e \notin H$:

$$\hat{f}_e(H) = \sum_R f_e(R)U_H(R)\mu(R) = \sum_{R|e\notin R} (f_e(R)U_H(R)\mu(R) + f_e(R\oplus e)U_H(R\oplus e)\mu(R\oplus e))$$

$$= \sum_{R|e\notin R} \left(f_e(R)U_H(R)\mu(R) + f_e(R\oplus e)U_H(R)\mu(R)\frac{p_k}{q_k}\right) = \sum_{R|e\notin R} \left(f_e(R) + \frac{p_k}{q_k} f_e(R\oplus e)\right) U_H(R)\mu(R) = 0$$

since $U_H(R) = U_H(R \oplus e)$ and $\mu(R \oplus e) = \frac{p_k}{q_k}\mu(R)$ when $e \notin H$.

The case where $e \in H$ is more technically involved. First, let's note the following properties

$$U_H(R) = \begin{cases} -\frac{p_k}{q_k} U_H(R\oplus e), & \text{if } e \notin R \\ -\frac{q_k}{p_k} U_H(R\oplus e), & \text{if } e \in R \end{cases}$$

and

$$\mu(R) = \begin{cases} \frac{q_k}{p_k} \mu(R\oplus e), & \text{if } e \notin R \\ \frac{p_k}{q_k} \mu(R\oplus e), & \text{if } e \in R \end{cases}$$

In particular

$$U_H(R)\mu(R) = -U_H(R\oplus e)\mu(R\oplus e)$$

To verify, that $\sum_R \hat{f}_e(R)U_H(R)\mu(R) = \hat{f}(H)$, let's split the summation into two regimes based on the value of $f(R)$.

- $f(R) = 1$:

$$\hat{f}_e(R)U_H(R)\mu(R) = q_k f(R)U_H(R)\mu(R) + q_k f(R\oplus e)U_H(R\oplus e)\mu(R\oplus e)$$

- $f(R) = 0$:

$$\hat{f}_e(R)U_H(R)\mu(R) = p_k f(R)U_H(R)\mu(R) + p_k f(R\oplus e)U_H(R\oplus e)\mu(R\oplus e)$$

Hence, summing over all possible R's, we obtain precisely the expansion of $\hat{f}(H)$.

□

2.7 Weights of large subgraphs are negligible

Lemma 2.3 (influence corresponds to e-dependent Fourier weights)**.** *Let e be an edge of arity k, then*

$$q_k \cdot p_k \cdot I_e = \sum_{H|e\in H} \hat{f}^2(H)$$

Proof. From Parseval's identity

$$\|f_e\|_2^2 = \sum_{H|e\in H} \hat{f}^2(H)$$

however

$$\|f_e\|_2^2 = \sum_R f_e(R)^2\mu(R)$$

the corresponding summand for each R is non-zero iff R is flipping. The total measure of such sets is I_e. Moreover the measure of flipping sets containing and not containing e is $p_k \cdot I_e$ and $q_k \cdot I_e$ respectively. Therefore

$$\sum_R f_e(R)^2\mu(R) = I_e \cdot p_k \cdot q_k^2 + I_e \cdot q_k \cdot (-p_k)^2 = I_e \cdot p_k \cdot q_k$$

$\square$

By summing this equality over every edge $\sum_k p_k \cdot q_k \cdot I^{(k)} = \sum_H \hat{f}^2(H)|H|$. In particular, this implies

$$\sum_{H||H|\geq L} \hat{f}^2(H) \leq \sum_k q_k \cdot p_k \cdot I^{(k)}/L \leq \frac{1}{L}\sum_k p_k \cdot I^{(k)}$$

where the weighted sum of influences is bounded by a constant under our assumptions.

2.8 Modest (hyper)graphs

Let $L, c_1, c_2 \in \mathbb{R}$ be fixed parameters. Together, they define a family of *modest* graphs. A graph $\mathcal{H}$ is modest if:

- $|\mathcal{H}| \leq L$
- $c_1 \leq E(\mathcal{H}) \leq c_2$
- $\mathcal{H}$ is balanced

for certain bounds L, c_1, c_2.

In particular we are only interested in the *edge-stucture* of the modest graphs, hence they have a constant number of vertices and there's finitely many of them.

3 Lemmas on random (hyper)graphs

Since the underlying property is symmetric, coefficients corresponding to isomorphic subgraphs are equal. In particular it's useful to define an aggregating notation for the basis functions.

For a graph S let $\Theta(S)$ be the set of all n-vertex graphs isomorphic to S, i.e. the orbit of S. Let

$$V_S := \sum_{H\in\Theta(S)} U_H$$

For $e \in H$, let

$$V_{H,e} := \sum_{R\in\Theta(H)|e\in R} U_R$$

3.1 Concentration bound on $V_{H,e}$

Note that $E(V_{H_e}) = 0$. Let's show a concentration bound on $|V_{H,e}|$.

Lemma 3.1 (Concentration bound on $V_{H,e}$)**.**
Let $\widetilde{H}$ be non-empty subgraph of H minimizing $D(\widetilde{H})$, then

$$\mathbb{P}\left[|V_{H,e}| \geq \lambda \cdot \frac{\sqrt{|\Theta(H)|}}{D(\widetilde{H})^{1/4}}\right] \leq c \cdot \frac{\lambda^{-4}}{n^{|e|}}$$

Proof. First let's bound the 4th moment of $V_{H,e}$

$$E(V_{H,e}^4) = E\left(\left(\sum_{R\in\Theta(H)|e\in R} U_R\right)^4\right) \leq c \cdot E\left(\sum_S \frac{|\Theta(S)|}{n^{|e|}} \sum_L \prod_{R\in L} |U_R|\right)$$

where S is the union of four copies of H containing edge e and L is a quadruple of copies of H s.t. their union is S. $\frac{|\Theta(S)|}{n^{|e|}}$ roughly corresponds to the number of copies of S containing e.

Let S be a union of four copies of H corresponding to a quadruple L. For an edge $e \in S$, let c_e be the number of copies of H covering e.

$$E\left(\prod_{R\in L} U_R\right) = E\prod_{e\in S}(U_e^{c_e}) = \prod_{e\in S}\left(p_{|e|}\cdot\left(-\sqrt{\frac{q_{|e|}}{p_{|e|}}}\right)^{c_e} + q_{|e|}\cdot\left(\sqrt{\frac{p_{|e|}}{q_{|e|}}}\right)^{c_e}\right)$$

Note, that for this product to be non-zero, every edge in S needs to be at least doubly covered. Therefore, the dominant terms are $p_{|e|}\cdot\left(-\sqrt{\frac{q_{|e|}}{p_{|e|}}}\right)^{c_e}$. Hence

$$E\left(\prod_{R\in L} U_R\right) \sim \prod_k \frac{p_k^{|S^{(k)}|}}{\left(\sqrt{p_k}\right)^{4|H^{(k)}|}}$$

Observe, that the double sum $\sum_S\sum_L$ is bounded by a constant depending on H. With $E(S) = |\Theta(H)|\prod_k p_k^{|H^{(k)}}$, we obtain

$$E\left(V^4\right) \leq c \cdot \prod_k p_k^{-2|H^{(k)}} \max_S \frac{E(S)}{n^{|e|}} \leq c \cdot \left(\frac{1}{\mathcal{P}_H}\right)^2 \max_S \frac{D(S)}{n^{|e|}}$$

since $E(S) \leq D(S)$.

To complete the proof, we'll show that the maximum is obtained with S being union of two copies of H whose intersection is exactly $\widetilde{H}$. From the calculations above, we know that every edge in S needs to be covered at least twice, but for convenience let's loosen up this restriction. Let x be a vertex or an edge, let's introduce a function which counts how many times x is covered by a copy of H

$$\Phi(x) := \begin{cases} 1/2, \text{ if } x \text{ is covered exactly once} \\ 1, \text{ if } x \text{ is covered at least twice} \\ 0, \text{ else} \end{cases}$$

For a graph R, let $\widetilde{v}(R) := \sum_{v\in v(R)}\Phi(v)$ and $\widetilde{e}(R) := \sum_{e\in R}\Phi(e)$. Then, the function $\widetilde{D}(R) := n^{\widetilde{v}(R)}\prod_k p_k^{\widetilde{e}(R^{(k)})}$ is identical with $D(R)$ when every edge of R is covered at least twice, however it's easier to maximize.

Let F be the union of four copies of H. By adding the copies one by one, the first two of them increase the value of $\widetilde{D}$ by exactly $\sqrt{D(H)}$, while the other two increase it by at most $\sqrt{\frac{D(H)}{D(\widetilde{H})}}$. Hence $\max_S D(S) \leq \frac{D(H)^2}{D(\widetilde{H})}$ and

$$E\left(V^4\right) \leq c \cdot \left(\frac{1}{\mathcal{P}_H}\right)^2 \frac{D(H)^2}{D\left(\widetilde{H}\right) \cdot n^{|e|}} = c \cdot \left(\frac{1}{\mathcal{P}_H}\right)^2 \frac{n^{2|v(H)|}\mathcal{P}_H^2}{n^{|v(\widetilde{H})|}\mathcal{P}_{\widetilde{H}}} \leq c \cdot \frac{|\Theta(H)|^2}{n^{|e|} \cdot D\left(\widetilde{H}\right)}$$

where the last inequality follows from $|\Theta(H) \sim c \cdot n^{|v(H)|}$ and the lemma follows from the Markov's inequality. $\square$

This lemma is particularly useful to use due to it's immediate corollary. Let $\chi = \chi\left\{|V_{H,e}| > \frac{\sqrt{|\Theta(H)|}}{D(\widetilde{H})^{1/4}}\right\}$, then

$$\int |V_{H,e}| \cdot \chi = E\left(|V_{H,e}| \cdot \chi\right) \leq c \cdot \frac{\sqrt{|\Theta(H)|}}{D(\widetilde{H})^{1/4} \cdot n^{|e|}} \tag{1}$$

3.2 V_S expansion

Given a graph R, let X_R be a random variable counting the appearances of R in a random graph. Let $X_\emptyset = 1$. V_S can be expressed in terms of X_R, where $R \subseteq S$.

Lemma 3.2 (V_S expressed in terms of X_R's)**.**
For any graph S

$$V_S = \sqrt{\frac{1}{\mathcal{P}_S \mathcal{Q}_S}} \cdot E(S) \left(\sum_{R \subseteq S} (-1)^{|R|} \frac{X_R}{E(R)}\right)$$

Proof. For any fixed subgraph R, let Y_R be the indicator that R appeared in our random graph. Let H be a fixed copy of S and let $H' \subseteq H$ be maximal graphs such that $Y_{H'} = 1$. Then U_H can be expressed as

$$U_H = (-1)^{|H|} \sqrt{\frac{\mathcal{Q}_H}{\mathcal{P}_H}} \cdot (-1)^{|H|-|H'|} \frac{\mathcal{P}_{H \setminus H'}}{\mathcal{Q}_{H \setminus H'}}$$

The later can be further expanded into

$$(-1)^{|H|} \sqrt{\frac{\mathcal{Q}_H}{\mathcal{P}_H}} \cdot \left(\sum_{H' \subseteq H} (-1)^{|H|-|H'|} \frac{\mathcal{P}_{H \setminus H'}}{\mathcal{Q}_{H \setminus H'}} \sum_{H' \subseteq H'' \subseteq H} (-1)^{|H''|-|H'|} Y_{H''}\right)$$

by changing the order of summation, we get

$$U_H = \sqrt{\frac{\mathcal{Q}_H}{\mathcal{P}_H}} \cdot \sum_{H'' \subseteq H} (-1)^{|H''|} \sum_{H' \subseteq H''} \frac{\mathcal{P}_{H \setminus H'}}{\mathcal{Q}_{H \setminus H'}} Y_{H''}$$

Observe that due to symmetry, each subgraph isomorphic to H'' appears in exactly $\frac{|\Theta(H)|}{|\Theta(H'')|}$ subgraphs isomorphic to H. Then, by summing over $\Theta(S)$

$$V_S = \sqrt{\frac{\mathcal{Q}_H}{\mathcal{P}_H}} \cdot \sum_{H''\subseteq H} (-1)^{|H''|} \sum_{H'\subseteq H''} \frac{\mathcal{P}_{H\setminus H'}}{\mathcal{Q}_{H\setminus H'}} \frac{|\Theta(H)|}{|\Theta(H'')|} X_{H''}$$

$$= \sqrt{\frac{1}{\mathcal{P}_S\mathcal{Q}_S}} \cdot E(S) \left(\sum_{R\subseteq S} (-1)^{|R|} \frac{X_R}{E(R)} \sum_{H'\subseteq R} \mathcal{Q}_{H'}\mathcal{P}_{R\setminus H'} \right)$$

□

3.3 Variation of X_R

Lemma 3.3 (Variation of X_R)**.**
Let R be a fixed graph and let X_R be a random variable counting copies of R in our random graph. Assume that for every k, $p_k = o(1)$. Then

$$Var(X_R) = c \cdot \frac{E(R)^2}{\sum_{H\subseteq R} E(H)}$$

Proof. $Var(X_R)$ can be expressed as $\sum_{X,Y} E(XY) - E(X)E(Y)$ where the sum is over the indicators of each possible copy of R in a random graph. When the intersection of edges of X and Y is empty, the corresponding summand is 0, otherwise $E(X)E(Y) = o(E(XY))$. Therefore

$$\sum_{X,Y} E(XY) - E(X)E(Y) = c \cdot \sum_{X,Y|X\cap Y\neq\emptyset} E(XY)$$

By partitioning this sum according to the isomorphism type of the intersection, we get

$$Var(X_R) = c \cdot \sum_{H\subseteq R} \frac{E(R)^2}{E(H)}$$

□

4 Approximation of the characteristic function

To show the main theorem, we will use an auxilary lemma, which will be proven later.

Lemma 4.1 (Main lemma)**.**
Let $\mathcal{A}$ be a symmetric, monotone family of graphs on n vertices and f be its characteristic function. Assume $\sum_k p_k \cdot I^{(k)(\chi_{\mathcal{A}})} \leq c$. Then for every $\varepsilon > 0$ there exist constants L, c_1, c_2 s.t. for sufficiently large n

$$\sum_{S \text{ is not modest}} \hat{f}^2(S) \leq \varepsilon$$

4.1 Proof of the main theorem

Let $S_1, S_2, \ldots, S_l$ be a list of all modest (according to the parameters from Lemma 4.1) graphs. Let $\mathcal{C}_S$ be the set of all n-vertex graphs s.t. the union of all modest graphs in each of them is isomorphic to S. For any graph $\mathcal{H} \in \mathcal{C}_S$, we will refer to S as its template.

4.1.1 Non-modest graphs don't matter

Let $g_1 = \sum_S \hat{f}(S) V_S$, where the sum is over modest graphs only be the first approximation of our function. In particular, due to the Main lemma, $\|f - g_1\| \leq \varepsilon_1$ for arbitrarily chosen ε_1.

4.1.2 Large templates don't matter

Let $\mathcal{C}_1$ be the union of all C_S, s.t. $X_{S_i} \geq lE(S_i)/\varepsilon_2$ for some i. Let

$$g_2(H) = \begin{cases} 1, \text{ if } H \in \mathcal{C}_1 \\ g_1(H), \text{ otherwise} \end{cases}$$

Then by Markov's inequality and union-bound, $\|f - g_2\| \leq \varepsilon_1 + \sqrt{\varepsilon_2}$ for arbitrarily chosen ε_2.

4.1.3 There's finitely many interesting templates

Since the expected number of copies of S_i is bounded for any H with $g_2(H)$ not identically equal to 1, there is a finite number of such templates. Let M be an upper bound on the count of admissible templates.

This enables us to exclude templates with small measure. Let $\mathcal{C}_2$ be the union of C_S not in $\mathcal{C}_1$ s.t. $\mu(\mathcal{C}_S) \leq \varepsilon_3/M$. Then $\mu(\mathcal{C}_2) \leq \varepsilon_3$.

Let $\mathcal{C}$ be the union of the remaining sets $\mathcal{C}_S$ not in $\mathcal{C}_1, \mathcal{C}_2$.

Note that any graph S s.t. $\mathcal{C}_S \subset \mathcal{C}$ must have bounded size, as $\mathcal{C}_S \not\subset \mathcal{C}_1$. Moreover, since S is a union of modest graphs and $\mu(\mathcal{C}_S) \leq E(S)$, $E(S)$ is bounded by a constant from both sides. Combining it with the fact that S is a union of modest (balanced) graphs, it implies that S is balanced as well.

4.1.4 Template-constant approximation

In the previous steps we've approximated our original function using only graphs with not-too-large templates and non-negligible measure.

Now, we'll use templates to naturally split the remaining subgraphs into equivalence classes and define an approximation of f which is constant on each such class. Let

$$g_3(H) = \begin{cases} E\left(g_2 | \mathcal{C}_S\right), \text{ if } H \in \mathcal{C} \\ 1, \text{ otherwise} \end{cases}$$

It turns out, g_2 and g_3 are very close to each other. The proof of this lemma will be shown in a later section.

Lemma 4.2 (Approximations are almost template-constant)**.** *Let $\mathcal{C}_S \subset \mathcal{C}$. For any $\delta > 0$*

$$Pr\left[|g_2(T) - E(g_2|\mathcal{C}_S)| > \delta : \ T \in \mathcal{C}_S\right] \to 0$$

Since there are only M sets C_S in $\mathcal{C}$, for sufficiently small δ, $\|f - g_3\|_2^2 \leq \varepsilon_4/4$.

Let g_4 be a function, s.t. for every S is either constant 0 or 1 on $\mathcal{C}_S$, depending which approximates f better.Let h be the best template-constant L^2 approximation of f. It's easy to verify, that $h_{\mathcal{C}_S} = Pr\left[f(R) = 1 | \ R \in \mathcal{C}_S\right]$. Hence by summing over $\mathcal{C}_S$

$$\|f - g_4\|_2^2 \leq 4\|f - h\|_2^2 \leq 4\|f - g_3\|_2^2 = \varepsilon_4$$

4.2 Monotone approximation

The family of graphs that is characterized by g_4 is a natural candidate for the family whose existence is guaranteed by Main theorem. Unfortunately it may not be monotone, neither the size of its largest minimal element must be bounded.

Let S be *decisive*, if $Pr\left[f(T)=1 \mid T\in\mathcal{C}_S\right]\notin\left(\sqrt{\varepsilon_4},1-\sqrt{\varepsilon_4}\right)$ and $\mathcal{C}_3$ be the union of $\mathcal{C}_S$ where S is undecisive. From the previous calculation, $\mu(\mathcal{C}_3)\le\sqrt{\varepsilon_4}$. Let

$$g_5(H)=\begin{cases}1, \text{ if } H \text{ has a decisive subgraph } R \text{ with } g_4(R)=1\\ 0, \text{ otherwise}\end{cases}$$

Sets $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3$ are of small measure, therefore the alterations in the approximation of f caused by them are negligible. Moreover the following lemma guarantees that the approximation is not significantly altered for the decisive graphs as well. This lemma will be proven in the later section.

Lemma 4.3 (f is almost template-monotone)**.**
Let $R\subset T$ be graphs s.t. $\mathcal{C}_S,\mathcal{C}_T\subset\mathcal{C}\setminus\mathcal{C}_3$. Then

$$E(f|\mathcal{C}_T)>E(f|\mathcal{C}_R)-o(1)$$

Therefore, for carefully chosen $\varepsilon_1,\ldots,\varepsilon_4$, we have

$$\|f-g_5\|_2^2\le\varepsilon$$

and g_5 is the characteristic function of a family $\mathcal{B}$ guaranteed by the Main theorem.

By repeating this argument with a different choice of $\varepsilon_1,\ldots,\varepsilon_4$ we can define $\mathcal{C}_4$ s.t. for $\mathcal{C}_R\subset\mathcal{C}_4, E(f|\mathcal{C}_R)>1-\varepsilon$. Since both $E(R)$ and $|R|$ are bounded, any such R could be a graph M guaranteed by a Theorem 1.2. Let r be a specific copy of R and B_r be the space of graphs with r as a subgraph. Let $\mathcal{C}_r=B_r\cap\mathcal{C}_R$. Due to symmetry $E(f|\mathcal{C}_r)=E(f|\mathcal{C}_R)$. $\mathcal{A}$ is an increasing property, but being in $\mathcal{C}_r$ conditioned on being in B_r is a decreasing property, therefore

$$\mathcal{P}\left(\mathcal{A}\cap\mathcal{C}_r|B_r\right)\le\mathcal{P}\left(\mathcal{A}|B_r\right)\cdot\mathcal{P}\left(\mathcal{C}_r|B_r\right)$$

therefore

$$1-\varepsilon<E\left(f|\mathcal{C}_r\right)\le E\left(f|B_r\right)$$

5 Technical lemmas

Due to Lemma 2.3, subgraphs with large number of edges have negligible Fourier weights and can be neglected. For this reason we will narrow down our analysis in this section to graphs with the number of edges bounded by some constant independent of n.

5.1 Main lemma

5.1.1 Fourier weights of subgraphs with small expectations are negligible

First, let's show a bound on a Fourier weight of a subgraph H with edges of a single size k.

$$\hat{f}^2(H)=\left(\sum_R f(R)U_H(R)\mu(R)\right)^2=\left(\sum_{c=0}^{|H^{(k)}|}\binom{|H^{(k)}|}{c}f(R)\cdot\left(-\sqrt{\frac{q_k}{p_k}}\right)^c\cdot\sqrt{\frac{p_k}{q_k}}^{|H^{(k)}|-c}p_k^c\cdot q_k^{|H^{(k)}|-c}\right)^2$$

$$\leq \left(\sum_{c=0}^{|H^{(k)}|} \binom{|H^{(k)}|}{c} f(R) \cdot \sqrt{p_k}^{|H^{(k)}|} \right)^2 \leq \left(2^{|H^{(k)}|} \sqrt{p_k}^{|H^{(k)}|} \right)^2 = (4p)^{|H^{(k)}|}$$

where c in the summations correponds to the size of the intersection between R and H.

In order to generalize this bound to the non-uniform setting, such summation needs to be carried out over tuples corresponding to the sizes of intersection of different arities. Since each component of such tuple is independent of each other, this yields

$$\hat{f}^2(H) \leq \prod_k (4p_k)^{|H^{(k)}|} = 4^{|H|} \cdot \mathcal{P}_H \tag{2}$$

By summing over subgraphs isomorphic to H, i.e. its *orbit*, we obtain

$$\sum_{S \in \Theta(H)} \hat{f}^2(S) \leq cE(H) \tag{3}$$

Hence the graphs with small expectation have negligible Fourier weights.

In fact, even stronger lemma is true. It is enough for a graph to have a subgraph with small expectation for its Fourier weight to be negligible.

Lemma 5.1 (Subgraphs with small expectation causes negligible Fourier weight)**.** *Let H be a graph, then for every $H' \subseteq H$*

$$\sum_{S \in \Theta(H)} \hat{f}^2(S) \leq c \cdot \max_{R \subsetneq H'} \{E(H')/E(R)\}$$

where the expectation of an empty subgraph is 1*.*

Proof. Let R be a specific copy of H'. Consider a probability space, where R is fixed. Let $g(G) = f(G \cup R)$ be a function on this new space and let $\widetilde{\Theta}(S)$ denote S' orbit in the new, restricted space, i.e. under automorphisms which keep R fixed. Note, that both the vertices and edges of S and R might intersect, and the asymptotics of $\widetilde{\Theta}(S)$ might depend on these intersections. First, let's show that the Fourier expansions of f and g are closely related.

Lemma 5.2 (Fourier expansion on restricted domain)**.** *Let g be defined as above, then*

$$\hat{g}(G) = \sum_{R' \subseteq R} \hat{f}(R' \cup G) U_{R'}(R)$$

Proof. The proof will use an induction on $|R|$. First, let's assume that R is a single edge e of size k. Then, by definition

$$\hat{g}(G) = \sum_{T | e \notin T} f(T \cup e) U_G(T) \cdot \left(\mu(T) \frac{1}{q_k} \right)$$

The RHS of our desired equality is

$$\hat{f}(G) - \sqrt{\frac{q_k}{p_k}} \hat{f}(G \cup e) = \sum_W f(W) \left(U_G(W) - \sqrt{\frac{q_k}{p_k}} U_{G \cup e}(W) \right) \mu(W)$$

where the summands are non-zero only when $e \in W$. Using $U_{G \cup e}(M) = -\sqrt{\frac{q_k}{p_k}} U_G(W \setminus e)$ we get

$$\sum_{W | e \in W} f(W) \left(U_G(W) + \frac{q_k}{p_k} U_G(W \setminus e) \right) \mu(W)$$

$$= \sum_{W | e \in W} f(W) \left(U_G(W \setminus e) + \frac{q_k}{p_k} U_G(W \setminus e) \right) \cdot \left(\mu(W \setminus e) \frac{p_k}{q_k} \right) = \sum_{W | e \notin W} f(W \cup e) U_G(W) \mu(W) \frac{1}{q_k}$$

which settles the proof when $|R| = 1$.

When $|R| > 1$ pick an edge $e \in R$ of size k and define an auxilary function h by fixing all edges in $R \setminus e$, i.e. $g(G) = h(G \cup e) = f(G \cup R)$. We will use h as an intermediate step to apply the induction hypothesis twice - first by expressing the coefficients of g in terms of h and then by expressing the coefficients of h in terms of f.

$$\hat{g}(G) = \hat{h}(G) - \sqrt{\frac{q_k}{p_k}} \hat{h}(G \cup e) = \sum_{R' \subseteq R \setminus e} \hat{f}(R' \cup G) U_{R'}(G) - \sqrt{\frac{q_k}{p_k}} \sum_{R' \subseteq R \setminus e} \hat{f}(R' \cup G \cup e) U_{R'}(G \cup e)$$

$e \notin R'$, hence $U_{R'}(G \cup e) = -\frac{q_k}{p_k} U_{R' \cup e}(G \cup e)$ and we obtain the desired result. $\square$

Let's note the following implication of Parseval's identity. For any G

$$\sum_{S \in \Theta(G)} \hat{f}^2(S) \leq \sum_S \hat{f}^2(S) \leq 1 \Rightarrow |\hat{f}(G)| \leq \frac{1}{\sqrt{|\Theta(G)|}}$$

Let G be a edge-disjoint with R. The result of lemma 5.2 can be equivalently written as

$$-\hat{f}(R \cup G) U_R(R) = -\hat{g}(G) + \sum_{R' \subset R} \hat{f}(R' \cup G) U_{R'}(R)$$

combined with $U_{R'}(R) = \prod_k \left(-\sqrt{q_k/p_k} \right)^{|R'^{(k)}|} \leq \prod_k \sqrt{1/p_k}^{|R'^{(k)}|}$, this yields

$$\left| \frac{\hat{f}(R \cup G)}{\prod_k \sqrt{p_k}^{|R^{(k)}|}} \right| \leq c \left(\frac{1}{\sqrt{|\widetilde{\Theta}(G)|}} + \sum_{R' \subset R} \left| \frac{\hat{f}(R' \cup G)}{\prod_k \sqrt{p_k}^{|R'^{(k)}|}} \right| \right)$$

hence

$$\left| \frac{\hat{f}(R \cup G)}{\prod_k \sqrt{p_k}^{|R^{(k)}|}} \right| \leq c \left(\frac{1}{\sqrt{|\widetilde{\Theta}(G)|}} + \sum_{R' \subset R} \frac{1}{\prod_k \sqrt{p_k}^{|R'^{(k)}|} \sqrt{|\Theta(R' \cup G)|}} \right) \tag{4}$$

Let's observe, that for $R' \subseteq R$

$$|\Theta(R' \cup G)| \geq c \cdot |\Theta(R')| \cdot |\widetilde{\Theta}(G)| \tag{5}$$

Morover, when $v(R') \cap v(G) = v(R) \cap v(G)$, the opposite inequality also hold (with a different constant c)

$$|\Theta(R' \cup G)| \leq c \cdot |\Theta(R')| \cdot |\widetilde{\Theta}(G)| \tag{6}$$

Let G be s.t. $H = G \cup R$. Let's multiply both sides of (4) by $\sqrt{|\Theta(H)| \prod_k p_k^{|R|^{(k)}}}$. By applying (5) and (6) we obtain

$$\sqrt{\sum_{S \in \Theta(H)} \hat{f}^2(S)} \leq c \left(\sqrt{E(R)} + \sum_{R' \subset R} \sqrt{\frac{E(R)}{E(R')}} \right)$$

which after squaring both sides and $|R|$ being bounded yields the desired result. $\square$

5.1.2 Subgraphs with non-negligible weights are balanced

Let H be such, that $W = \sum_{R \in \Theta(H)} \hat{f}^2(R)$ is bounded away from 0. From the facts shown in the previous section, we know that such H must be of bounded size and with the expectancy bounded away from 0. To show the Main lemma, we will show that such H must also be modest. It is left to show that it's balanced and with expectation bounded from above.

Let's recall $\int f \cdot g = \sum \hat{f} \cdot \hat{g}$. Hence, for an edge e

$$\hat{f}(H) \int f_e \cdot V_{H,e} = \sum_{R \in \Theta(H) | e \in R} \hat{f}^2(H)$$

W can be alternatively expressed as

$$W = \sum_{R \in \Theta(H)} \sum_{e \in R} \frac{1}{|R|} \hat{f}^2(R) = \frac{1}{|H|} \sum_e \sum_{R \in \Theta(H) | e \in R} \hat{f}^2(H) = \frac{1}{|H|} \sum_e \hat{f}(H) \int f_e \cdot V_{H,e}$$

Let's note that $\hat{f}(H) = \sqrt{\frac{W}{|\Theta(H)|}}$, hence

$$W = \frac{1}{|H|} \sqrt{\frac{W}{|\Theta(H)|}} \sum_e \int f_e \cdot V_{H,e}$$

Let $\chi = \chi \left\{ |V_{H,e}| > \frac{\sqrt{|\Theta(H)|}}{D(\widetilde{H})^{1/4}} \right\}$, then the above can be rewritten as

$$W = \frac{1}{|H|} \sqrt{\frac{W}{|\Theta(H)|}} \sum_e \left(\int f_e V_{H,e} \cdot \chi + \int f_e V_{H,e} \cdot (1 - \chi) \right)$$

Since $|f_e| \leq 1$, the above can be bounded by

$$W \leq \frac{1}{|H|} \sqrt{\frac{W}{|\Theta(H)|}} \sum_e \left(\int |V_{H,e}| \cdot \chi + \int |f_e| \cdot \frac{\sqrt{|\Theta(H)|}}{D(\widetilde{H})^{1/4}} \right)$$

Due to the corollary of Concentration bound on $V_{H,e}$, the first term of the innermost sum contributes at most $c \frac{1}{|H|} \sqrt{\frac{W}{|\Theta(H)|}}$ to the total sum.

Let e be edge of arity k. Then, in order to bound the contribution of the second term, let's establish a bound on $\sum_e \int |f_e|$

$$\frac{1}{2} \|f_e\|_1 = \frac{1}{2} \sum_{H | H \text{ is flipping}} |f_e(H)| = \frac{1}{2} \left(p_k I_e \cdot |q_k| + q_k I_e \cdot |-p_k| \right) = q_k p_k I_e$$

Hence

$$\frac{1}{2} \sum_e \int |f_e| \leq \sum_k p_k I^{(k)}$$

Combining these two bounds, we obtain

$$W \leq c \sum_k p_k I^{(k)} \frac{\sqrt{|\Theta(H)|}}{D(\widetilde{H})^{1/4}}$$

Therefore

$$D(\widetilde{H}) \leq c \cdot \frac{\left(\sum_k p_k I^{(k)}\right)^4}{W^2} \tag{7}$$

Having established, that $D(\widetilde{H})$ is bounded by a constant from both above and below, we now wish to deduce that so is H. When $H = \widetilde{H}$ we are done, since not only $E(H)$ is bounded from above, but due to lemma 5.1 it is balanced, as it has no subgraphs with small expectation.

It is only left to show, that H is modest, when $\widetilde{H}$ is its proper subgraph. If there are multiple candidates for $\widetilde{H}$ i.e. multiple subgraphs with the same asymptotic order of the magnitude, we can choose the one which is smallest by inclusion. Hence $\widetilde{H}$ is strictly proper.

We will show, that $H \setminus \widetilde{H}$ is balanced, and so is H as a union of two balanced graphs. Let S be a specific copy of $\widetilde{H}$, then

$$g(R) = f(R \cup S)$$

i.e. g correponds to f when S is fixed (its edges are always included). Due to lemma 5.2

$$\hat{g}(R) = \sum_{R' \subseteq S} \hat{f}(R \cup S') U_{S'}(S)$$

combining with the fact, that $|\hat{f}(G)| \leq 1/\sqrt{|\Theta(G)|}$ and $U_{S'}(S) \leq \sqrt{1/\mathcal{P}_{S'}}$, this yields

$$|\hat{f}(R \cup S') U_{S'}(S)| \leq \frac{1}{\sqrt{|\Theta(R \cup S')| \mathcal{P}_{S'}}} \leq \frac{1}{\sqrt{E(S')}}$$

Let's note, that since S is strictly balanced, the term corresponding to $S' = S$ in lemma 5.2 dominates the sum. Hence

$$\hat{g}(H \setminus S) = c \cdot \frac{|\hat{f}(H)|}{\sqrt{\mathcal{P}_S}} \tag{8}$$

Note, that $|\widetilde{\Theta}(H \setminus \widetilde{H})| \cdot |\Theta(\widetilde{H})| = c|\Theta(H)|$ and let $W' = \sum_{G \in \widetilde{\Theta}(H \setminus S)} \hat{g}^2(G)$. Then

$$W' = |\widetilde{\Theta}(H \setminus \widetilde{H})| \hat{g}^2(H \setminus S) = c \cdot \frac{|\Theta(H)|}{|\Theta(\widetilde{H})|} \cdot \frac{\hat{f}^2(H)}{\mathcal{P}_S} = c \frac{W}{E(\widetilde{H})}$$

hence W' is also a constant.

To complete the argument, we will show that $\sum_k q_k p_k I^{(k)}(g)$ is at most polylogarithmic. Then, we can iterate our argument on $H \setminus \widetilde{H}$ to a point, where the graph minimizing D is the whole graph and use the following observation:

Let's recall, that every p_k is a rational power of n. If there exists a bounded graph H, where $E(H)$ is a constant, then there exists a universal constant c s.t. for every other graph G of bounded size and any integer k, for large enough n

$$E(G) < \log(n)^m \implies E(G) < c$$

Therefore $E(H)$ must be bounded by a constant, since it is a disjoint union of modest graphs. Hence, it also must be balanced and therefore modest.

To finish the proof, let's show that $\sum_k q_k p_k I^{(k)}(g)$ is indeed at most polylogarithmic. Let's recall, that $\sum_k p_k I^{(k)}(g)$ can be interpreted as the expected number of edges, which *flip* g. This can be alternatively phrased, as the expected number of edges which *flip* f, conditioned on the appearance of S. Let C be a random variable counting the number of flipping edges and $\mathcal{S}$ be the event that S appears in the random graph. Let X_H be the random variable counting the appearances of copies of H in our random graph. Then

$$E(C|\mathcal{S}) = \sum_i E\left(C|X_{\widetilde{H}} = i\right) \mathbb{P}\left(X_{\widetilde{H}} = i|\mathcal{S}\right) = \sum_i E\left(C|X_{\widetilde{H}} = i\right) \mathbb{P}\left(X_{\widetilde{H}} = i\right) \frac{\mathbb{P}\left(\mathcal{S}|X_{\widetilde{H}=i}\right)}{\mathbb{P}(\mathcal{S})}$$

$$= \sum_i E\left(C|X_{\widetilde{H}} = i\right) \mathbb{P}\left(X_{\widetilde{H}} = i\right) \frac{i}{E\left(\widetilde{H}\right)}$$

Due to the following lemma, summands with $i > \log n$ can be neglected and therefore conditioning on $\mathcal{S}$ increases $E(C)$ by at most logarithmic factor.

Lemma 5.3 (There are few copies of a graph with low expectation)**.**
Let H be strictly balanced s.t. $E(H)$ is bounded by a constant. Then for every k, the probability of having more than $\log n$ copies of H in our random graph is asymptotically less than n^{-k}.

Proof. Let's split the proof into two subcases, depending on the number of copies of H intersecting other copies.

First, let's assume that at most B copies of H intersect other copies, where B is sufficiently large integer determined in the second case. Let $R = \log(n) - B$. Since H is strictly balanced, of bounded size and with expectation bounded by constant, the distribution of X_H converges to Poisson distribution with $E(H)$ as its parameter, e.g. see [3], theorem 1.21. Therefore, probability of X_H being at least R can be bounded by $c \cdot \frac{E(H)^R}{R!}$ which is asymptotically less than $\frac{1}{2}n^{-k}$.

The remaining case is where there are at least B copies H intersecting other copies. Each such instance can be characterized as having at least one of finitely many graphs as a subgraph i.e. union of at least B and at most $2B$ copies of H. Since H is strictly balanced, the expected number of occurences of each such witness is negligible. In particular, in our setting it's a negative, rational power of n. Since the list of potential witnesses is finite, for sufficiently large B, the probability of the second case can also be bounded by $\frac{1}{2}n^{-k}$ which completes the proof. □

5.2 Monotone lemma

Let $\mathcal{C}, \mathcal{C}_3$ be defined as in the section 4. Now, we will proof the lemma which allowed us to *monotonize* our family of graphs.

Lemma 4.3 (f is almost template-monotone)**.**
Let $R \subset T$ be graphs s.t. $\mathcal{C}_S, \mathcal{C}_T \subset \mathcal{C} \setminus \mathcal{C}_3$. Then

$$E(f|\mathcal{C}_T) > E(f|\mathcal{C}_R) - o(1)$$

Proof. Let's recall, that both R and T are balanced and $E(R), E(T), \mu(R)$ and $\mu(T)$ are all bounded by a constant both from above and below.

Let r and t be specific copies of R and T. Let B_r be the space of all graphs with r as a subgraph and let $\mathcal{C}_r = \mathcal{C}_R \cap B_r$. Let B_t and $\mathcal{C}_t$ be defined analogously.

Due to symmetry of $\mathcal{C}_R$

$$E(f|\mathcal{C}_R) = E(f|\mathcal{C}_r)$$

and

$$E(f|\mathcal{C}_T) = E(f|\mathcal{C}_t)$$

Let's define $\sigma : \mathcal{C}_t \to \mathcal{C}_r$ be an injective and measure preserving with respect to the conditional measure of $\mathcal{C}_t$ and its image. That is, for any $G \in \mathcal{C}_t$, $\mu_1(G) = \mu_2(\sigma(G))$, where μ_1 and μ_2 are conditional measures on $\mathcal{C}_t$ and $\mathcal{C}_r$ respectively. Moreover σ will be such, that

$$f(\sigma(G)) = 1 \implies f(G) = 1$$

Then, to show our lemma it will suffice to show

$$E(f|\sigma(\mathcal{C}_t)) \geq E(f|\mathcal{C}_r) - o(1) \tag{9}$$

Such σ can be defined as an operation which for $G \in \mathcal{C}_r$ deletes the edges in $t \setminus r$. Let $W_1 = \sigma(\mathcal{C}_t)$ and $W_2 = \mathcal{C}_r \setminus W_1$. For a graph $G \in \mathcal{C}_r$ let

$$\alpha(G) = \mathbb{P}\left[\pi(G) \in W_2\right] \tag{10}$$

where π is a permutation of the vertices leaving the r fixed. Note, that the edges "follow" their original endpoints after relabeling, as the goal is to determine the proportion of W_2 in $\mathcal{C}_r$. Let

$$a = \frac{\mu(W_2)}{\mu(\mathcal{C}_r)}$$

and let μ_r be the conditional measure on $\mathcal{C}_r$. Then

$$a = \int_{\mathcal{C}_r} \alpha \; d\mu_r$$

On the other hand, the proportion of W_1 to $\mu(\mathcal{C}_r)$ can be expressed as

$$1 - a = \frac{\mu(\sigma(\mathcal{C}_t))}{\mu(\mathcal{C}_r)} = \frac{\mu(\mathcal{C}_t)}{\mathcal{P}_{t \setminus r} \cdot \mu(\mathcal{C}_r)} = \frac{\mu(\mathcal{C}_T)}{\Theta(T)} \frac{\Theta(R)}{\mu(\mathcal{C}_R)} \frac{\mathcal{P}_R}{\mathcal{P}_T} = \frac{\mu(\mathcal{C}_T)E(R)}{\mu(\mathcal{C}_R)E(T)}$$

Therefore a is asymptotically a constant. Then

$$E(f|W_1) = \frac{\int_{\mathcal{C}_r} f \cdot (1 - \alpha) \; d\mu_r}{1 - a}$$

Let $d = (1 - a) - (1 - \alpha)$ be the difference between $1 - \alpha$ and its mean. Then, the above can be rewritten as

$$\frac{\int_{\mathcal{C}_r} f \cdot (1 - a + d) \; d\mu_r}{1 - a} = \int_{\mathcal{C}_r} f \; d\mu_r - \frac{\int_{\mathcal{C}_r} f \cdot d \; d\mu_r}{1 - a}$$

and due to Cauchy-Schwarz inequality, it can be bounded from below by

$$E(f|\mathcal{C}_r) - \frac{\sqrt{\int_{\mathcal{C}_r} f \cdot |d|^2 \; d\mu_r}}{1 - a} = E(f|\mathcal{C}_r) - \frac{\sqrt{Var(\alpha)}}{1 - a}$$

Combined with the following lemma, this will imply (9).

Lemma 5.4 (α is highly concentrated around mean)**.**
Let $\alpha(G)$ be defined as in (10) and let $G \in \mathcal{C}_r$ be chosen according to the measure μ_r. Then

$$Var(\alpha) = o(1)$$

Proof. It is more conveniant to work in B_r instead of $\mathcal{C}_r$ due to the convenience of the product measure. Note, that the proportion of graphs in B_r with edges in $t \setminus r$ is negligible and we will ignore them in future calculations. Let's note

$$\frac{\mu(\mathcal{C}_r)}{\mu(B_r)} = \frac{\mu(\mathcal{C}_R)}{\mathcal{P}_R \cdot |\Theta(R)|} = \frac{\mu(\mathcal{C}_R)}{E(R)}$$

therefore C_r constitutes a non-negligible fraction of B_r. Therefore it suffices to show that $Var(\alpha|B_r) = o(1)$. For reminder of this proof, we will work in the space B_r.

First, let's define the notion of problematic sets. Let $v_1, v_2, \ldots, v_k$ be the vertices in $t \setminus r$. For an ordered tuple of vertices $x = (x_1, x_2, \ldots, x_k)$ disjoint with $V(r)$, let π_x be a permutation which keeps r fixed and maps x_i to v_i. Then x is problematic if there exists a set of veritces and edges $y \in G \setminus (x \cup r)$ (not necessarily a graph) s.t. $x \cup y$ is a graph and the graph $\pi_x(G)[V(t) \cup V(y)] \cup t$ is a union of modest graphs other than t, i.e. the graph induceed on the vertices of r and y with the addition of the edges of t creates a template larger than t. We call such y an *extension* of x.

Let $X(G) = n_k\alpha(G)$, where $n_k = \frac{n!}{(n-k)!}$ be the number of problematic sets in G. Let $x_1, x_2, \ldots, x_{n_k}$ be the indicator variables of the event that their corresponding k-tuples are problematic. Therefore $X = \sum_i x_i$.

If $E(X) = o(n^k)$, then $E(\alpha) = o(1)$ and since $\alpha \in (0, 1)$, $Var(\alpha) = o(1)$. Therefore we can assume, that $E(X) = \Omega(n^k)$ and try to show, that $Var(X) = o(n^{2k})$. We have

$$Var(X) = \sum_i \sum_j (E(x_i x_j) - E(x_i)E(x_j))$$

For $i \neq j$, let $x_i \circ x_j$ be the random variable indicating that there exist edge disjoint extensions of sets x_i, x_j. As edge disjoint events are independent

$$E(x_i \circ x_j) \leq E(x_i)E(x_j)$$

Let $x_i \diamond x_j = x_i x_j - x_i \circ x_j$. Then

$$Var(X) = \sum_i \left(E(x_i^2) - E(x_i)^2\right) + \sum_{i \neq j} (E(x_i x_j) - E(x_i)E(x_j))$$

$$\leq \sum_i Var(x_i) + \sum_{i \neq j} (E(x_i x_j) - E(x_i \circ x_j)) = \sum_{i \neq j} E(x_i \diamond x_j) + o(n^{2k})$$

Before dealing with the terms $E(x_i \diamond x_j)$, let's define some useful notation. First, let's extend the definition of the function D to any set of vertices and edges (not necessarily graphs) in order to apply it to the extensions. For the reminder of the proof, when refering to some sets having larger (smaller) expectancy than others, it is meant in the asymptotics of D.

An extension y is nice if

- it is minimal extension
- $D(y) = \Theta(1)$
- For any $z \subset y$ s.t. $z \cup t$ is a graph, $D(z) = \Omega(1)$.

Let $x_i \star x_j$ be the indicator of event $x_i \diamond x_j$ when there exists a nice extension. Such definition is helpful, since when $x_i \diamond x_j = 1$, there exist some minimal extansion, and the probability of a subestension z with $D(z) = o(1)$ is marginal. Therefore

$$\sum_{i \neq j} E\left(x_i \diamond x_j - x_i \star x_j\right) = o(n^{2k})$$

and it suffices to show that

$$E\left(\sum_{i \neq j} x_i \star x_j\right) = o(n^{2k})$$

For an extension y, let $Cl(y)$ be the graph whose edges whose edges are edges in y. We'll show, that for any x and its nice extension y, any subgraph $z \subseteq Cl(y) \setminus r$, $D(z) = \omega(1)$.

For simplicity of the proof of the above observation, let y be a nice extension of x, where $x = V(t) \setminus V(r)$, as the argument easily translates to the other scenario. Since y is minimal, there exist a modest graph S s.t. $Cl(y) \subseteq S \subseteq y \cup t$. Then S can be deocomposed into three disjoint sets: $S \cap r$, $S \cap (t \setminus r)$ and $S \cap (y \setminus t)$.

Since y is nice $D(S \cap (y \setminus t)) = \Omega(1)$. Additionally, $D(S \cap (t \setminus r)) = \Omega(1)$ as otherwise its union with r would have negligible expectancy. Therefore for $S \cap r$, $D(S \cap r) = \mathcal{O}(1)$ and since r is modest, $D(S \cap r) = \Theta(1)$. Now, if there exist $z \subset Cl(y) \setminus r$ s.t. $D(z) = \Theta(1)$, then $z \cup (Cl(y) \setminus r)$ is modest which contradicts the fact that the union of modest graphs in our graph is r, which concludes the proof.

From the proof above, we can also note, that every nice extension has bounded size (since its closure is a subgraph of a bounded graph). Therefore $x \cup y$ can take on a finite number of isomorphism types $G_1, \ldots, G_d$.

Let $\widetilde{E}(G)$ be the expectancy of copies isomorphic to G (disjoint with r) in the space B_r. For each isomorphism type G_i mentioned above, there exist its copy g, such that $g \cup t$ is union of modest graphs. Therefore $E(g \cup t) = \mathcal{O}(1)$. On the other hand, $E(g \cup t) = \Omega\left(E(t)\widetilde{E}(g)/n^k\right)$. Combining these two bounds gives

$$\widetilde{E}(G_i) = \mathcal{O}(n^k)$$

Now, we are ready to bound $\sum_{i \neq j} E(x_i \star x_j)$ and finish the proof. Let x_i, x_j be such, that $x_i \star x_j = 1$ with x_i corresponding to a nice extension of type R_i and x_j to type R_j. Let H be the isomorphism type of their intersection. As H is proper subgraph of both R_i and R_j, we've previously shown, that $\widetilde{E}(H) = \omega(1)$ to our sum. Hence events of such type contribute at most $\widetilde{E}(R_i)\widetilde{E}(R_j)/\widetilde{E}(H) = o(n^{2k})$. There are finitely many distinct combinations of R_i, R_j and H, which yields the desired bound. □

□

5.3 Approximations are almost template-constant

Lemma 4.2 (Approximations are almost template-constant)**.** *Let $\mathcal{C}_S \subset \mathcal{C}$. For any $\delta > 0$*

$$Pr\left[|g_2(T) - E(g_2|\mathcal{C}_S)| > \delta : \ T \in \mathcal{C}_S\right] \to 0$$

Proof. Recalling the definition of g_2, the fact that there's a finite number of modest graphs and (2), it is sufficient to show the following

Let S be a modest graph and G be such, that $\mathcal{C}_G \subseteq \mathcal{C}$. Then for any $\delta > 0$

$$\mathbb{P}\left[|V_S(T) - E(V_S|\mathcal{C}_G)| > \frac{\delta}{\sqrt{\mathcal{P}_S}} : T \in \mathcal{C}_G\right] \to 0 \tag{11}$$

To do so, we will use Lemma 3.2, but first, let's show that for any $R \subseteq S$, $\frac{X_R}{E(R)}$ is almost constant i.e. X_R is either bounded by a constant or $Var(X_R|\mathcal{C}_G) = o(E(R)^2)$.

If R is modest, since $\mathcal{C}_G \subset \mathcal{C}$, then X_R is bounded by a constant and so is our fraction.

Recalling the Lemma 3.3, X_R's variation is asymptotically $\frac{E(R)^2}{\sum_{H\subseteq R} E(H)}$. This formula gives only the asymptotic formula for unconditioned variance, but since the event we are conditioning on (being in $\mathcal{C}_G$) has probability bounded away from 0 by a constant, the variance conditioned on such event can not be asymptotically larger than the unconditioned variance. Therefore, if R has no modest subgraphs, then each summand is $o(E(R)^2)$ and so is the variance. From now on assume that R is non-modest subgraph of a modest graph, which has a modest subgraph.

Let g be a specific copy of G. Let's define B_g to be the space of random graphs containing g as a subgraph. As in the past sections, let $\widetilde{E}(H)$ be the expectancy of copies of H and $\widetilde{\Theta}(H)$ the orbit of H in the space B_g. Let $\mathcal{C}_g = \mathcal{C}_G \cap B_g$. Due to symmetry, both the expectancy and variance of X_R conditioned on being in C_g are the same as when conditioned on being in $\mathcal{C}_G$.

Let's recall, that for any graph in C_g, g is the union of all modest graphs. Since R has a modest subgraph, then any copy of R is a union of some $g' \subset g$ and a copy of some graph T_i, from the family of finitely many templates $T_1, \ldots, T_k$. Therefore, we can define X_T as the random variable counting the appearances of copies of T_i. Naturally, $X_T = X_R$. Since g is the union of modest graph, for any $H \subseteq T_i$, $\widetilde{E}(H) = \omega(1)$.

We'll show that $Var(X_T) = o(E(X_T)^2)$, when conditioned on C_g. Due to the following computation, it is sufficient to show it, when conditioned on B_g. First, let's compute $\mu(\mathcal{C}_G)$

$$\mu(\mathcal{C}_G) = |\Theta(G)| \cdot \mathcal{P}_G \cdot \frac{\mu(C_g)}{\mu(B_g)}$$

this can be rearanged as

$$\frac{\mu(\mathcal{C}_G \cap B_g)}{\mu(B_g)} = \frac{\mu(\mathcal{C}_G)}{E(G)}$$

which shows, that C_g is a non-negligible part of B_g and the variance conditioned on C_g can not be significantly larger than when conditioned on B_g.

Let $x_1, \ldots, x_k$ be the indicator variables for each possible copy of some T_i, i.e. $X_T = \sum_i x_i$. Then, variance can be expressed as

$$Var(X_T|B_g) = \sum_{i,j} E(x_i x_j) - E(x_i)E(x_j)$$

Note, that summands corresponding to events caused by dijsoint set of edges are 0, therefore

$$Var(X_T|B_g) \leq \sum_{\substack{i,j \\ x_j,\, x_i \text{ not independent}}} E(x_i x_j) = \sum_{\substack{i,j \\ x_j \text{ and } x_i \text{ not independent}}} \frac{E(x_i)E(x_j)}{\mathcal{P}_H}$$

where H is the intersection of subgraphs corresponding to the events x_i, x_j. But since each H is non-empty, $\mathcal{P}_H = \omega(1)$ and $Var(X_T|B_g) = o(E(X_T)^2)$.

Let $c = \sum_{H \in \Omega(S)} \hat{f}^2(H)$. Then $|\hat{f}(S)| = \frac{c}{\sqrt{|\Theta(S)|}}$. Using Lemma 3.2

$$|\hat{f}(S)|V_S = c \cdot \sqrt{\frac{\mathcal{Q}_S}{\mathcal{P}_S}} \cdot \frac{E(S)}{\sqrt{|\Theta(S)|}} \left(\sum_{R \subseteq S} (-1)^{|R|} \frac{X_R}{E(R)} \right)$$

$$= c \cdot \sqrt{\mathcal{Q}_S \mathcal{P}_S} \cdot \sqrt{|\Theta(S)|} \left(\sum_{R \subseteq S} (-1)^{|R|} \frac{X_R}{E(R)} \right) = c \cdot \sqrt{\mathcal{Q}_S E(S)} \left(\sum_{R \subseteq S} (-1)^{|R|} \frac{X_R}{E(R)} \right)$$

Therefore

$$V_S = c \cdot \sqrt{\frac{\mathcal{Q}_S}{\mathcal{P}_S}} \cdot E(S) \left(\sum_{R \subseteq S} (-1)^{|R|} \frac{X_R}{E(R)} \right)$$

which combined with $Var(X_R|\mathcal{C}_G) = o(E(X_R)^2)$ which we showed above gives (11).

Having showed (11), let's recall (2) i.e.

$$\hat{f}^2(S) \leq 4^{|S|} \cdot \mathcal{P}_S$$

combining these two and the definition of g_2 this completes the proof of the lemma as well as the main result. □

6 Applicable theorems

In this section, we will derive "applicable" versions of the main theorem - inspired by Theorem 2.4 from [2]. First let's' recall Theorem 1.2

Theorem 1.2 (Special graph)**.**
Let $0 < \alpha < 1$. There exist functions $B(\varepsilon, c), b_1(\varepsilon, c), b_2(\varepsilon, c)$ s.t. for all $c > 0$, any n and any monotone symmetric family of graphs $\mathcal{A}$ on n vertices the following holds: If there exists a sequence (p_k) where each p_k is proportional to a rational power of n, $\sum_k p_k \cdot I^{(k)} \leq c$ and $\alpha < \mu(\mathcal{A}) < 1 - \alpha$, then for every $\varepsilon > 0$ there exists a graph M with the following properties:

- *M is balanced*
- $b_1 < E(M) < b_2$
- $|M| \leq B$
- *Let $\mathcal{P}[\mathcal{A}|M]$ denote the probability that a random graph belongs to $\mathcal{A}$ conditioned on the appearance of a* specific *copy of M. Then*

$$\mathcal{P}[\mathcal{A}|M] \geq 1 - \varepsilon$$

Since the property $\mathcal{A}$ is symmetric, instead of conditioning on the appearance of a particular copy of M, we may first sample random graph and then add a random copy of M. With this observation we can restate Theorem 1.2 as follows

Theorem 6.1.
Let $\mathcal{A}$ be a graph property with a coarse threshold i.e. there exists $\varepsilon > 0, \alpha > 0$ and $\mathcal{M}$ such that for every k, $p_k = \mathcal{M}(k)$ is either 0 or proportional to a rational power of n and

$$\alpha < \mathcal{P}\left(\widehat{H}(n; (1-\varepsilon)\mathcal{M}) \in \mathcal{A}\right) < \mathcal{P}\left(\widehat{H}(n; \mathcal{M}) \in \mathcal{A}\right) < 1 - 2\alpha$$

then there exists graph M with no more than $B(\varepsilon, \alpha)$ edges such that

$$\mathcal{P}\left(\widehat{H}(n; \mathcal{M}) \cup M^* \in \mathcal{A}\right) > 1 - \alpha$$

i.e. inclusion of a random copy of M to a random graph significantly boosts the probability of $\mathcal{A}$. Morever M is balanced and $E(M) = \Theta(1)$.

The existence of such $\varepsilon > 0$ guarantees that $\sum_k p_k^* \cdot I^{(k)}$ is bounded for some $(1-\varepsilon)p_k \leq p_k^* \leq p_k$ and the rest follows from Theorem 1.2.

To further simplify the statement of the theorem, let's recall the definition of sharp threshold and Russo-Margulis lemma. In order for a property to have coarse threshold, there needs to be infinitely many n's, where uniform multiplicative increment of $\mathcal{M}$ by ε does not induce the property almost surely, for any $\varepsilon > 0$.

Additionally, instead of working with randomly sampled graph, we can pick an arbitrary graph G satisfying desired properties.

Theorem 6.2 (Applicable theorem)**.**
Let $\mathcal{A}$ be a graph property. Let $\alpha > 0, \mathcal{M} := \mathcal{M}_n$ such that every $p_k := \mathcal{M}_n(k)$ is either 0 or proportional to a rational power of n and

$$\alpha < \mathcal{P}\left(\widehat{H}(n; \mathcal{M}) \in \mathcal{A}\right) < 1 - 3\alpha$$

If $\mathcal{A}$ has coarse threshold, then there exists a fixed balanced graph M with $E(M) = \Theta(1)$, $\varepsilon > 0$ and an infinite series of n's, and for each such n a graph G on n vertices such that the following holds

$$\mathcal{P}\left(G \cup M^* \in \mathcal{A}\right) > 1 - \alpha$$

$$\mathcal{P}\left(G \cup \widehat{H}(n; \varepsilon\mathcal{M}) \in \mathcal{A}\right) < 1 - 2\alpha$$

i.e. inclusion of a random copy of M boosts the probability of $\mathcal{A}$ more than adding additional ε proportion of random edges.

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