

Two-colorability of random non-uniform hypergraphs

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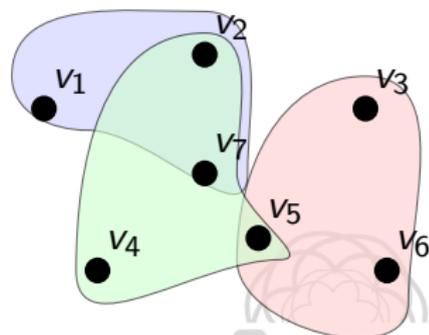
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A hypergraph $\mathcal{H} = (V, E)$ is a pair consisting of a set of vertices V and a set of (hyper)edges $E \subseteq \mathcal{P}(V)$.



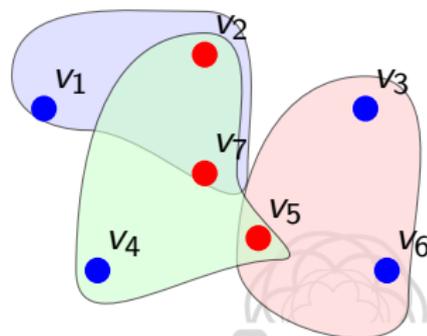
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Proper (vertex) 2-coloring

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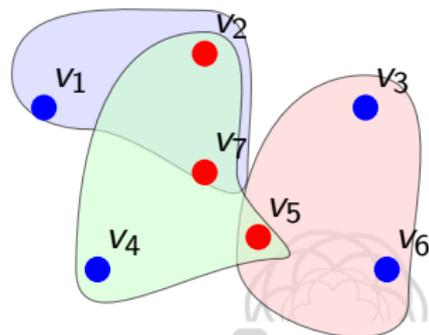
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2-coloring is hard

NP-complete for k -graphs, with $k \geq 3$.



Random hypergraphs

Random k -graphs (generalised Erdős–Rényi model)

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Random non-uniform hypergraphs

For a function $\mathcal{P} : \mathbb{N} \rightarrow [0, 1]$, let $\hat{H}(n; \mathcal{P})$ be a distribution of random hypergraphs, where each of $\binom{n}{k}$ possible edges is included independently with probability $\mathcal{P}(k)$.

2-coloring random k -graphs

- ▶ What is the maximum expected number of edges s.t. a random k -graph is 2-colorable w.h.p?



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A random k-graph on n vertices with at least $\frac{\ln 2}{2} \cdot 2^k \cdot n$ expected edges is a.a.s. not-2-colorable.



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With k large enough, a random k -graph with less than $\frac{\ln 2}{2} \cdot (2^k - 1 - \frac{1}{\ln 2}) \cdot n$ expected hyperedges is a.a.s. 2-colorable.

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The lower-bound was later improved by Coja-Oghlan and Zdeborova.

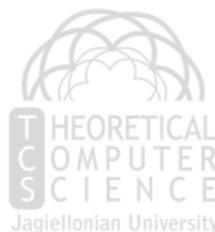
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Non-uniform upper-bound

Consider a distribution of random non-uniform hypergraphs $\hat{H}(n; \mathcal{P})$ s.t. the expected number of k -edges is $\lambda_k \cdot 2^k \cdot n$.

Suppose that $\sum_k \lambda_k \geq \frac{\ln 2}{2}$, then $\mathcal{H} \in \hat{H}(n; \mathcal{P})$ is a.a.s. not-2-colorable.

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Moreover, when $\sum_k \frac{\lambda_k}{1-2^{1-k}} < \frac{\ln 2}{2} (1 - \varepsilon)$ a random hypergraph is 2-colorable with **positive** probability.



Algorithmic 2-coloring

Algorithm introduced by Achlioptas, Krivelevich, Kim and Tetali (2002).

AKKT algorithm

do $n/2$ times:

if there exists a 3-tail or 2-tail e :

 pick any two uncolored vertices $x, y \in e$ and color x *RED*, y *BLUE*

else:

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Alg. k -uniform lower-bound (Achlioptas, Kim, Krivelevich, Tetali)

AKKT succeeds w.h.p on random k -graphs with no more than $c \cdot \frac{2^k}{k} \cdot n$ edges on average, with $c \leq \frac{1}{10}$ for $k \geq 40$.

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- ▶ In algorithmic setting, k -edge weighs $\frac{k}{2^k}$.

Auxiliary hypergraph

- ▶ Every partial coloring is equitable.



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AKKT's sufficient condition

If in every phase of the AKKT algorithm the expected number of new 3-tails is bounded by $1 - \varepsilon$ for $\varepsilon > 0$, then a.a.s. the algorithm succeeds in finding a proper 2-coloring.

Distribution of new 3-tails with a single edge size k

- ▶ Number of 3-tails at any phase is a Poisson variable.
- ▶ At any phase, with the expected number of k -edges equal $\lambda_k \cdot \frac{2^k}{k} \cdot n$, the Poisson's variable parameter is a linear function of λ_k .



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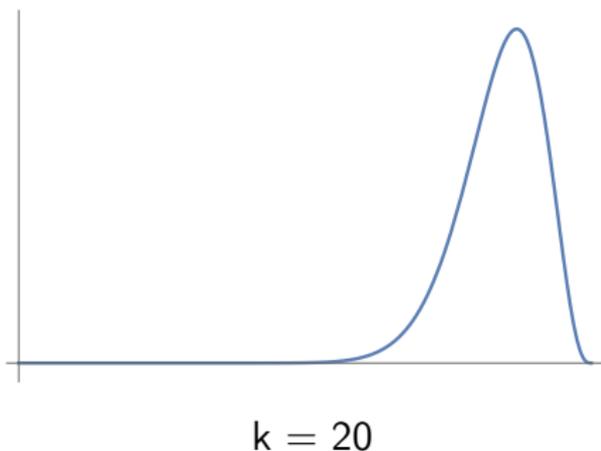
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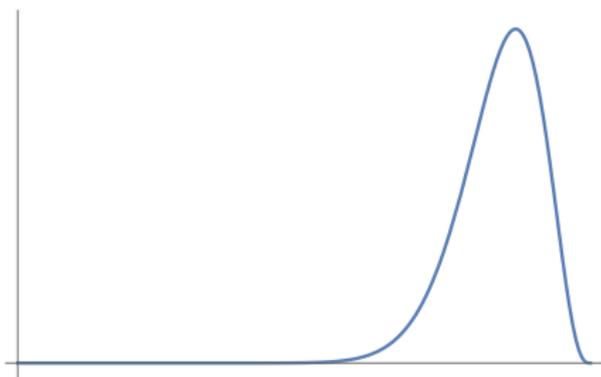
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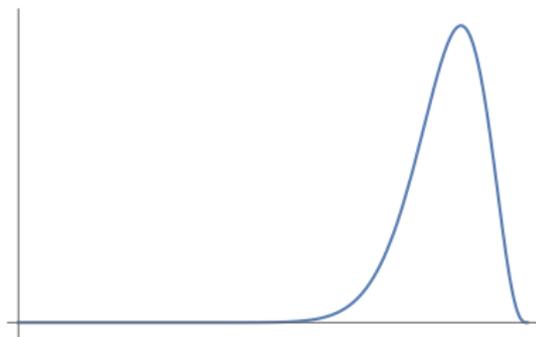


$k = 20$

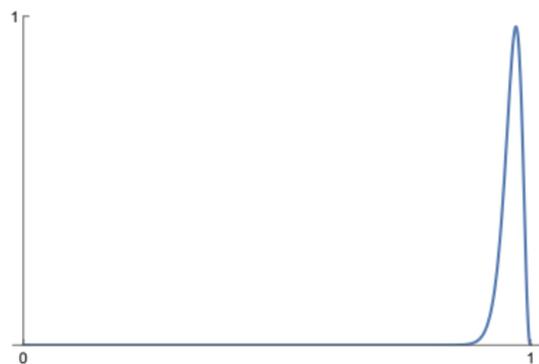


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Distribution of new 3-tails - we can do better

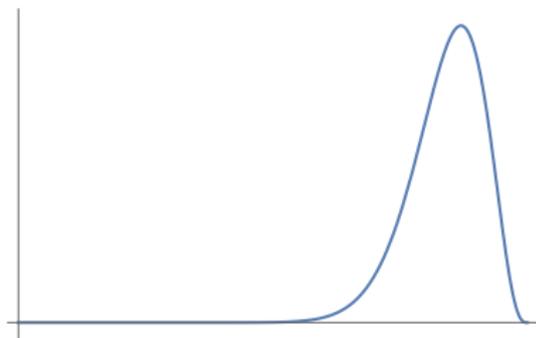


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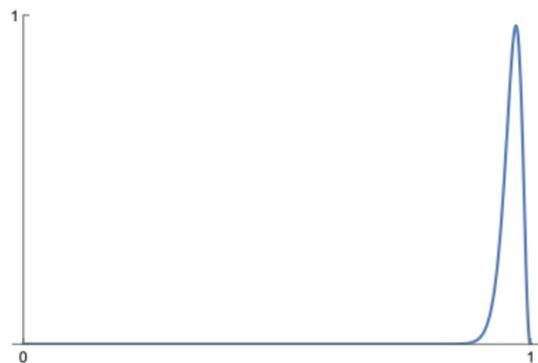


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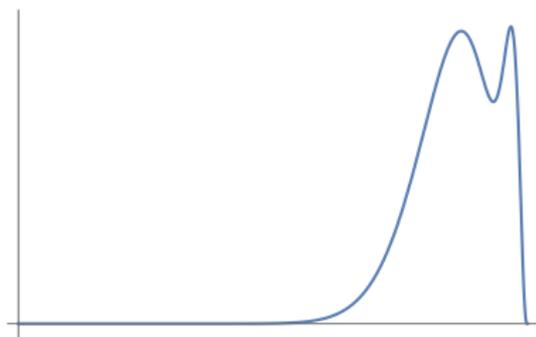
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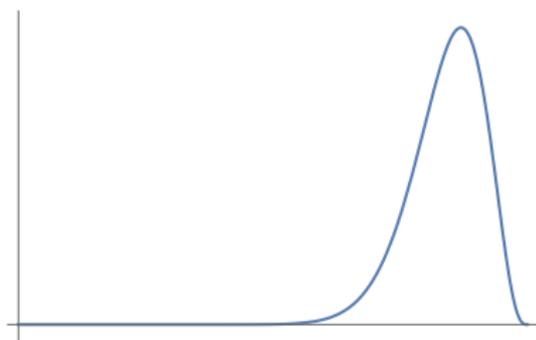


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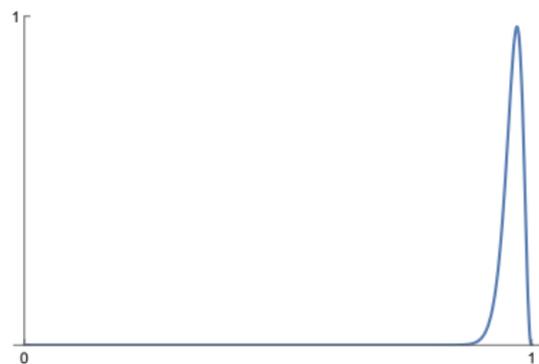


$k = 20$ and $k = 100$

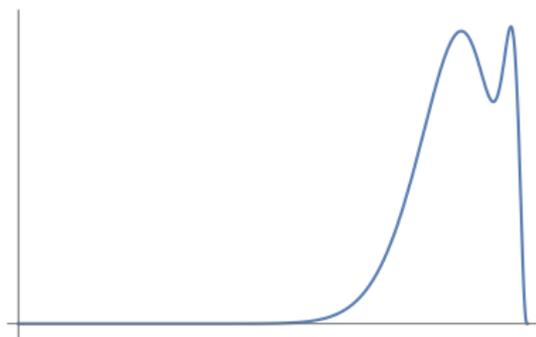
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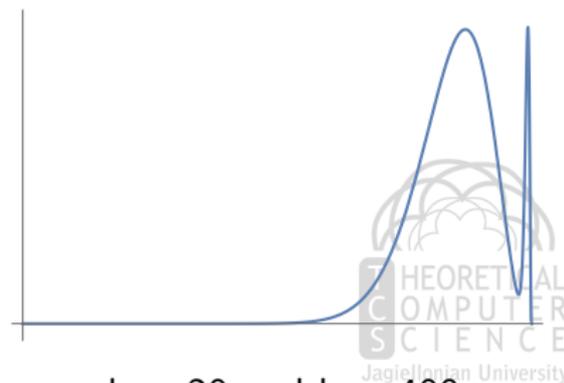
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Unbounded sum of lambdas

For every $c > 0$, there exists a random hypergraph model $\widehat{H}(n; \mathcal{P})$ s.t. the expected number of edges of size k is $\lambda_k \cdot \frac{2^k}{k} \cdot n$, for which

$$\sum_{k \in \text{supp}(\mathcal{M})} \lambda_k \geq c$$

and AKKT procedure succeeds a.a.s.

Algorithmic 2-coloring cont.

- ▶ We only need to bound a total amount of new 3-tails.



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AKKT aggregating condition

Consider a distribution of random hypergraphs $\widehat{H}(n; \mathcal{P})$ s.t. the expected number of k -edges is $\lambda_k \cdot \frac{2^k}{k} \cdot n$. Let

$$f(x) := \sum_{k \in \text{supp}(\mathcal{M})} \frac{\lambda_k}{k} \cdot x^k$$

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Then, for any $\varepsilon > 0$, the AKKT succeeds, when

$$\max_{x \in [0,1]} \left(\frac{16}{3} (1-x)^3 f^{(4)}(x) \right) < 1 - \varepsilon,$$

where $f^{(4)}$ is the fourth derivative of f .

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- ▶ The exponent 3 and fourth derivative come from considering 3-tails.

Thank you!

