Nonrepetitive sequences

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Ogólnopolska Konferencja Studentów Matematyki $\theta\beta\mathit{I}\iota\mathit{c}\mathbb{Z}\varepsilon$ 2025

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Introduction

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A typical problem in *combinatorics on words* asks 'Does exists an infinite sequance over some fixed alphabet, avoiding some *pattern*?'. This area of research was started by Axel Thue in 1906. He asked about the existance of *square-free* words.

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Clearly none nof them can be extended.

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- Given a square-free word w obtained with this procedure, the following substitions preserve this property:
  - $a \to abac$
  - $b \to bcac$
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The original proof of Thue used much more complex substitutions, however the main idea remained unchanged.

List coloring

Let's consider a slight generalization of the previous problem - list coloring. Previously, we worked in a setting with an alphabet which is identical for every position in our sequence. Now, let's assume that for every position $i \in \mathbb{N}$ we have a list L_i of admissable characters.

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In this setting, we want to find the smallest common size of lists guaranteeing a *proper* (square-free) character assignment. At first glance, it appears to be simpler problem, however in analogous problems such as *graph coloring* it's the opposite. Our situation is more complex and it is an open problem wheather list with three elements each are sufficient.

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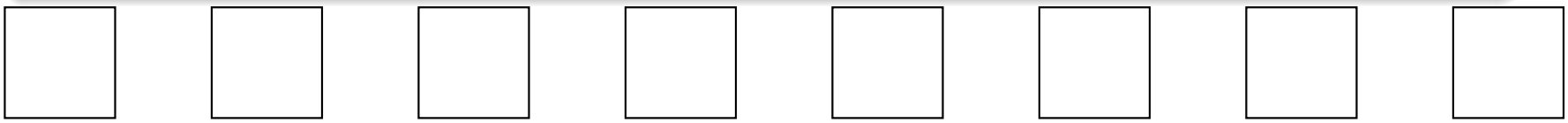

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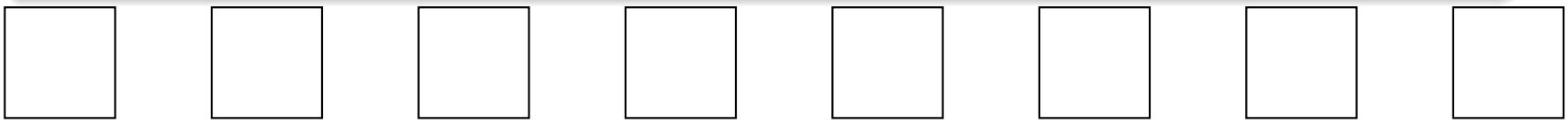

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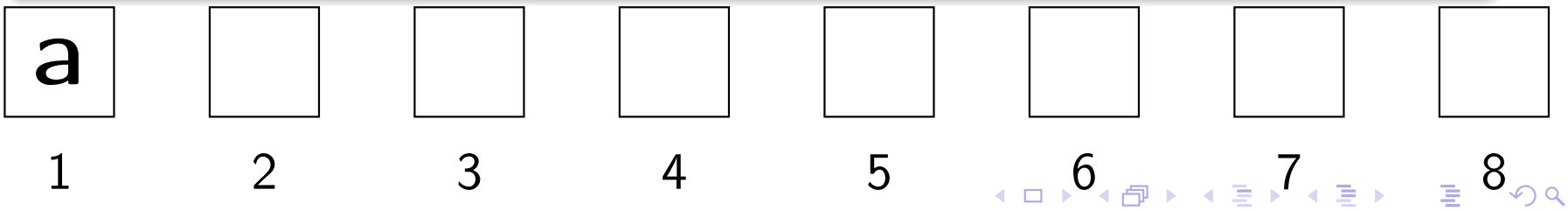

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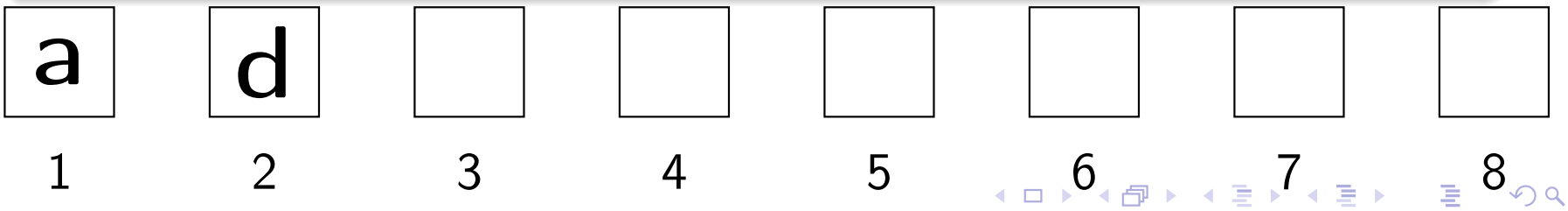

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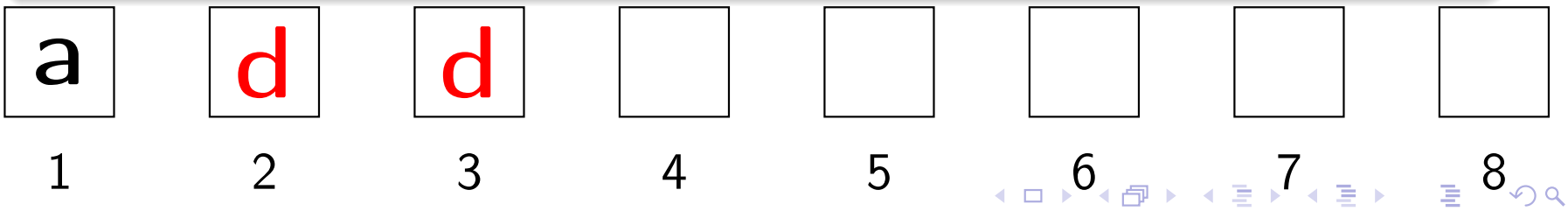

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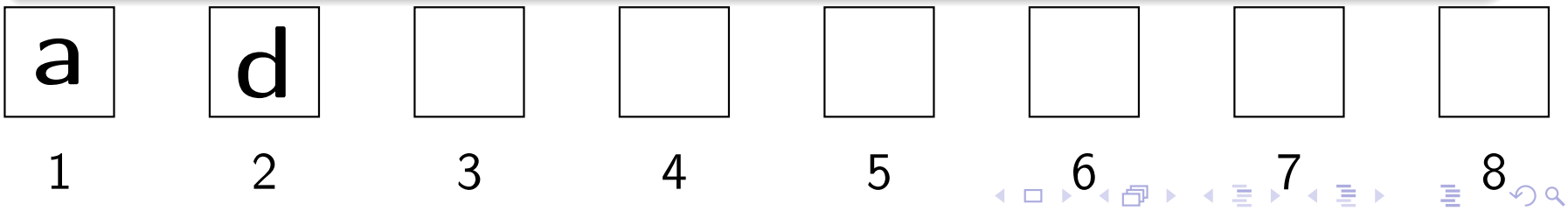

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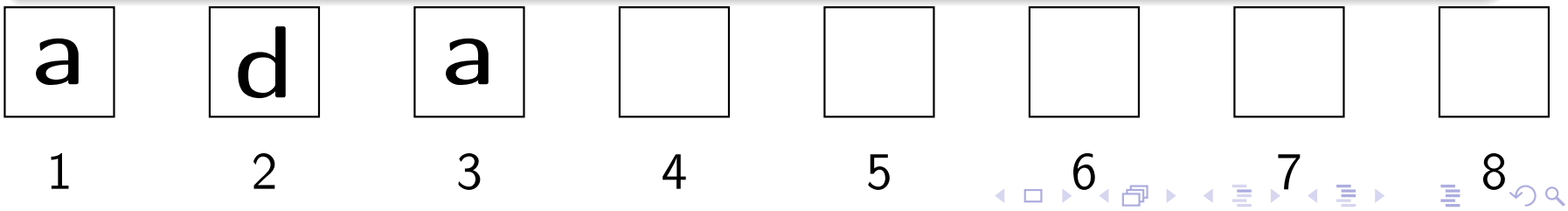

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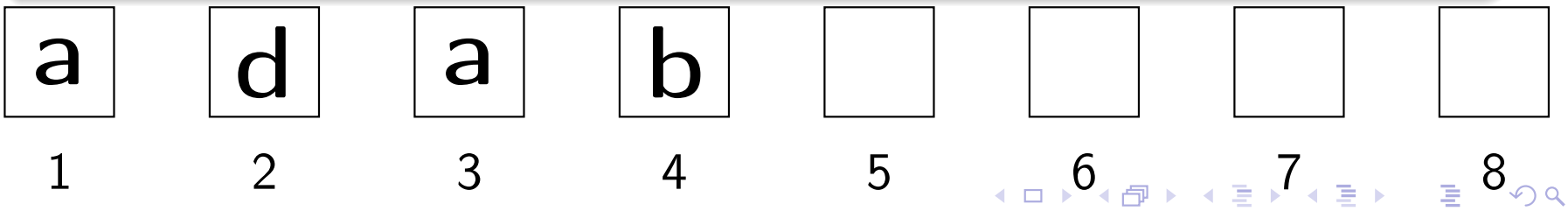

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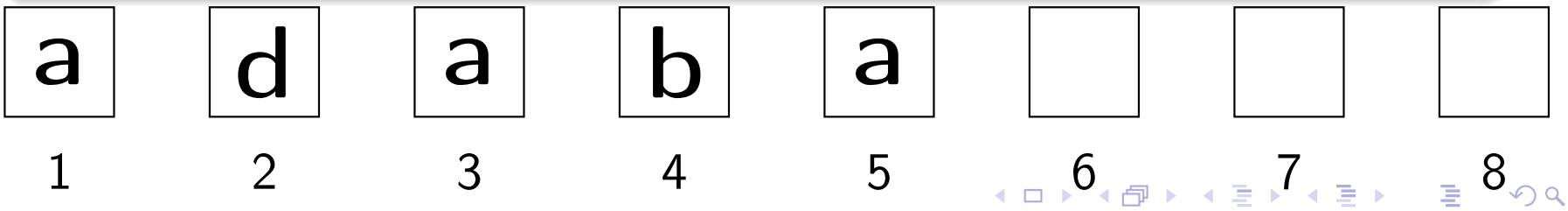

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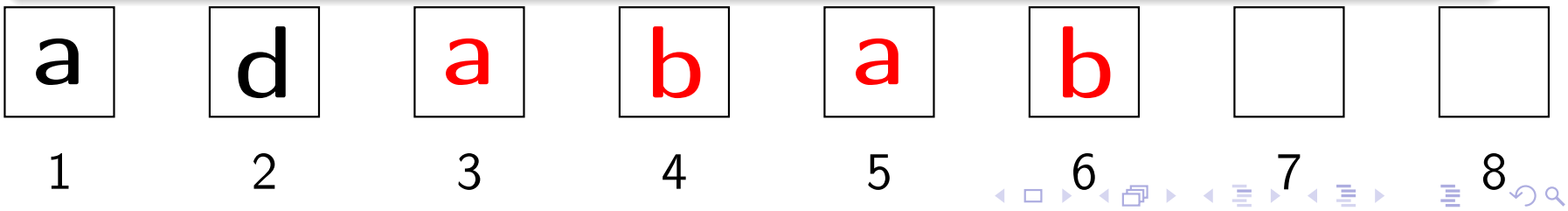

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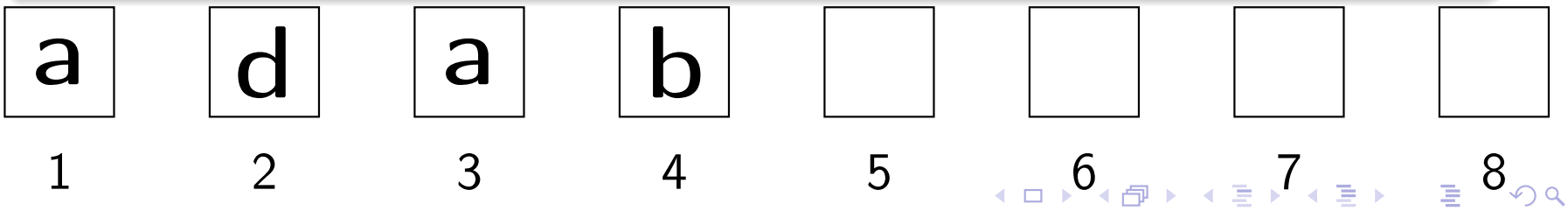

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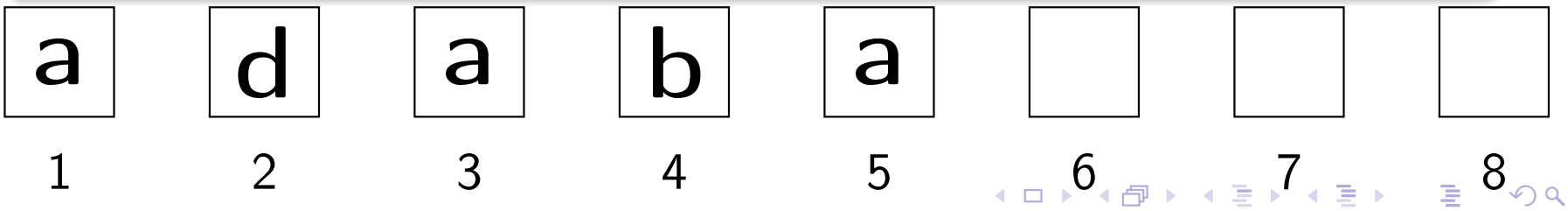

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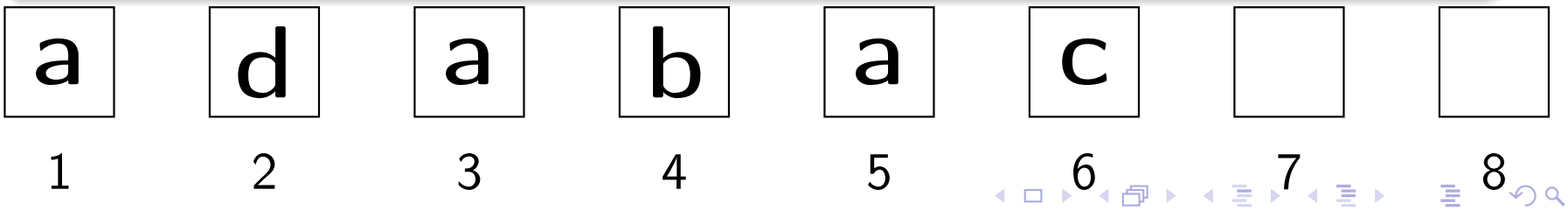

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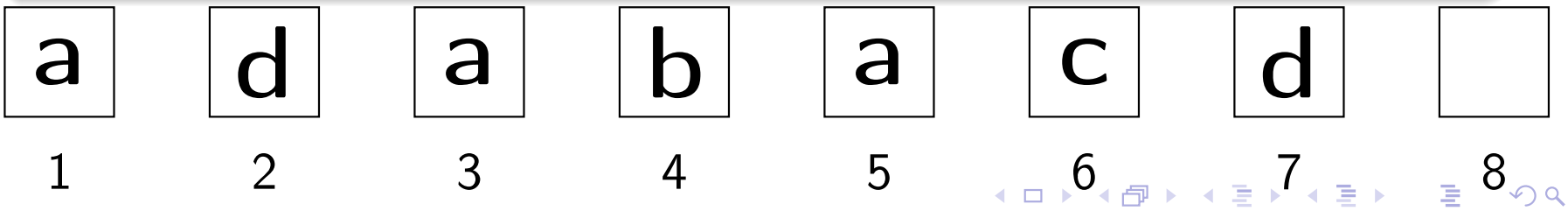

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a	d	a	b	a	c	d	c
1	2	3	4	5	6	7	8

Entropy compression argumen

With some intuition about how the algorithm works, we can attempt to prove it. The idea of the proof is based on the Moser and Tardos' proof of algorithmic Lovász Local Lemma. Their method was later called *Entropy compression argument* by Terrance Tao at his blog.

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- We will produce a *log* of each execution - an alternative represantation of each of 4^M runs.

Proof - cont.

Log

Let $c_1, c_2, \ldots, c_M$ be the choices of characters throughout the execution. Let's analyze the length of the word w throughout the execution.

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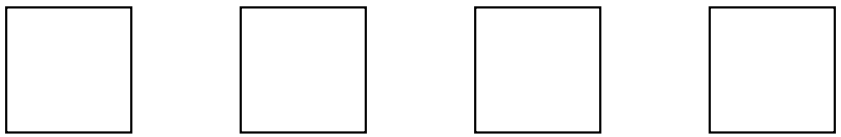

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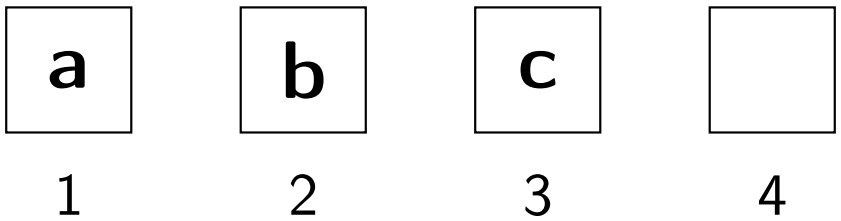

Diff	1	1	1	-1	1
Choice					

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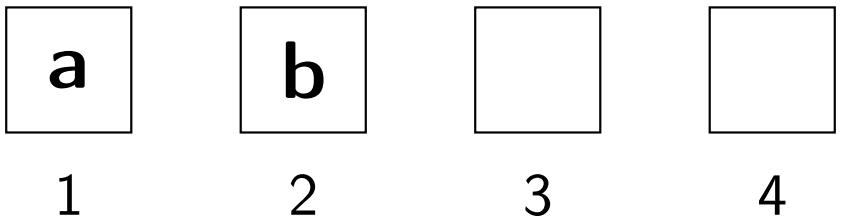

Diff	1	1	1	-1	1
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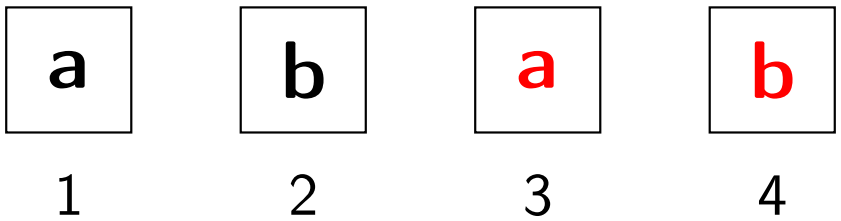

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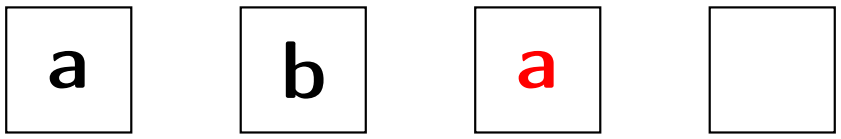
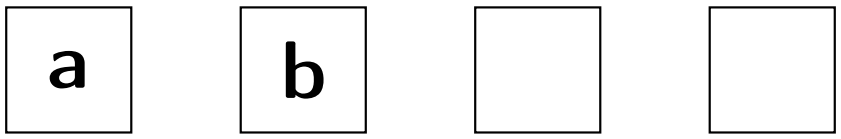


Diff	1	1	1	-1	1
Choice				b	c

Proof - cont.

Log

Let $c_1, c_2, \ldots, c_M$ be the choices of characters throughout the execution. Let's analyze the length of the word w throughout the execution. At each step, it can increase by one or decrease by arbitrary value s.t. w is non-empty. Let $d_1 = 1, d_2, \ldots, d_M$ be the sequence of the differences in length of w after each step of our algorithm. I.e. d_j is the difference between length of w after j-th step and $(j-1)$-th step. For a fixed execution, we will refer to a pair (d, w) - sequence of differences and the final word as a *log*.

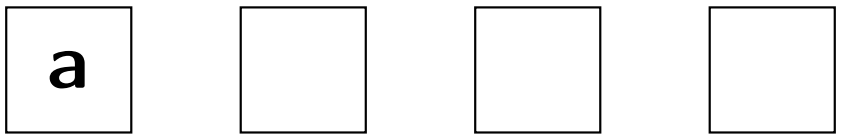


Diff	1	1	1	-1	1
Choice			a	b	c

Proof - cont.

Log

Let $c_1, c_2, \ldots, c_M$ be the choices of characters throughout the execution. Let's analyze the length of the word w throughout the execution. At each step, it can increase by one or decrease by arbitrary value s.t. w is non-empty. Let $d_1 = 1, d_2, \ldots, d_M$ be the sequence of the differences in length of w after each step of our algorithm. I.e. d_j is the difference between length of w after j-th step and $(j-1)$-th step. For a fixed execution, we will refer to a pair (d, w) - sequence of differences and the final word as a *log*.

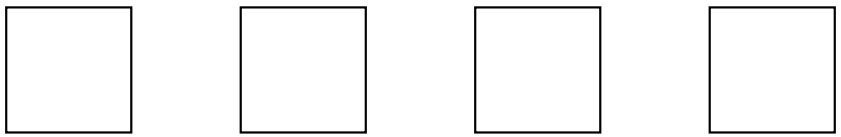


Diff	1	1	1	-1	1
Choice		b	a	b	c

Proof - cont.

Log

Let $c_1, c_2, \ldots, c_M$ be the choices of characters throughout the execution. Let's analyze the length of the word w throughout the execution. At each step, it can increase by one or decrease by arbitrary value s.t. w is non-empty. Let $d_1 = 1, d_2, \ldots, d_M$ be the sequence of the differences in length of w after each step of our algorithm. I.e. d_j is the difference between length of w after j-th step and $(j-1)$-th step. For a fixed execution, we will refer to a pair (d, w) - sequence of differences and the final word as a *log*.



1 2 3 4

Diff	1	1	1	-1	1
Choice	a	b	a	b	c

Proof - cont.

Since each execution can produce a log and every log corresponds to some execution, there is equal number of each of them.

Proof - cont.

Since each execution can produce a log and every log corresponds to some execution, there is equal number of each of them. The number of sequences d can be bounded by $n \cdot C_M$, where C_M is the M-th Catalan number. Hence the number of logs can be bounded by $4^n \cdot n \cdot C_M$. Let's note that $C_M = o(4^M)$. Hence:

$$4^M \leq n \cdot C_M \cdot 4^n$$

Proof - cont.

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$$4^M \leq n \cdot C_M \cdot 4^n = o(4^M)$$

Hence there exists a valid assignment regardless of the choice of the lists. □